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Massless and Massive Monopoles Carrying Nonabelian Magnetic Charges[1]

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ABSTRACT

The properties of BPS monopoles carrying nonabelian magnetic charges are investigated by following the behavior of the moduli space of solutions as the Higgs field is varied from a value giving a purely abelian symmetry breaking to one that leaves a nonabelian subgroup of the gauge symmetry unbroken. As the limit of nonabelian unbroken symmetry is reached, some of the fundamental abelian monopoles remain massive but acquire nonabelian magnetic charges. The BPS mass formula indicates that others should because massless in this limit. These do not correspond to distinct solitons, but instead are manifested as a "nonabelian cloud" surrounding the massive monopoles, with their position and phase degrees of freedom being transformed into parameters characterizing the cloud.

1 Introduction

Magnetic monopoles have long been the object of great interest. This is due in part to their role as predicted, but as yet undiscovered, particles in all grand unified theories. Beyond this, however, the monopoles of spontaneously broken gauge theories are of interest as examples of particles, arising from classical soliton solutions, that are in a sense complementary to the elementary quanta of the theory. It is particularly striking that this soliton-quanta complementarity mirrors the magnetic-electric duality of Maxwell's equations. This connection is made more concrete in the conjecture by

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Montonen and Olive [1] that certain theories may possess an exact magnetic-electric duality that exchanges solitons with elementary quanta and weak with strong coupling.

In this talk I will describe some research, done in collaboration with Kimyeong Lee and Piljin Yi [2], concerning monopoles that carry nonabelian magnetic charges; i.e., those whose long-range magnetic field transforms nontrivially under an unbroken nonabelian subgroup of the gauge symmetry of the theory. Just as the elementary quanta carrying nonabelian electric-type charges give rise to phenomena that are not seen with purely abelian charges, one might expect nonabelian magnetic charges to display interesting new features. Indeed, past investigations have discovered some curious properties associated with the long-range fields of such monopoles. For example, attempts to obtain chromodyons [3], objects with both electric and magnetic nonabelian charges, by applying time-dependent global gauge rotations are frustrated by topological obstructions [4] to the definition of such rotations in the presence of a nonabelian magnetic charge. Also, it has been shown [5] that the large-distance behavior of their Coulomb magnetic field can lead to instabilities in monopoles with more than a minimal nonabelian magnetic charge. New issues concerning these objects are raised by the duality conjecture. One, with which I will be particularly concerned in this talk, is the nature of the objects that are the magnetic counterparts of the massless nonabelian gauge bosons. Duality suggests that these should be massless, but it is not at all clear how one would obtain a zero energy soliton.

As I will describe below, our strategy was to start with a theory whose gauge symmetry is spontaneously broken to a purely abelian subgroup, and then to follow the behavior of the classical monopole solutions as the asymptotic Higgs field is varied to one of the special values that correspond to a nonabelian unbroken symmetry. To avoid the pathologies associated with the long-range behavior of nonabelian magnetic fields, we focused on systems, generally containing more than one monopole, whose total magnetic charge was purely abelian [6]. Throughout we worked in the Bogomolny-Prasad-Sommerfield (BPS) limit [7], with an adjoint representation Higgs field Φ. In addition, we made extensive use of the moduli space approximation [8], in which the low energy dynamics of interacting monopoles is reduced to that of a small number of collective coordinates.

The remainder of this talk is organized as follows. In Sec. 2, I review some of the properties of monopole and multimonopole solutions with both $SU(2)$ and larger gauge groups. The moduli space approximation is described in Sec. 3. In Sec. 4, I illustrate the behavior of monopoles as one goes from an abelian to a nonabelian unbroken symmetry, using an $SO(5)$ example for which it is possible to carry out explicit calculations. The extension to other gauge groups is discussed in Sec. 6. Section 7 contains come concluding remarks.

2 Multimonopole solutions with SU(2) and larger gauge groups

For an $SU(2)$ gauge theory whose symmetry is broken to $U(1)$ by a triplet Higgs field Φ, the asymptotic magnetic field is

$$B_i^a = \frac{g\hat{r}_i}{4\pi r^2}\frac{\Phi^a}{|\Phi|} \tag{1}$$

with g quantized in integer multiples of $4\pi/e$. The BPS solution carrying a single unit of magnetic charge is spherically symmetric and can be written in the form [7]

$$\begin{aligned} \Phi^a &= \hat{r}^a H(r) \\ A_i^a &= \epsilon_{aim}\hat{r}^m A(r)\,, \end{aligned} \tag{2}$$

where v is the asymptotic magnitude of the Higgs field and

$$\begin{aligned} A(r) &= \frac{v}{\sinh evr} - \frac{1}{er} \\ H(r) &= v\coth evr - \frac{1}{er}\,. \end{aligned} \tag{3}$$

The BPS solutions with $n > 1$ units of magnetic charge are all naturally interpreted as multimonopole solutions. Their energy is precisely n times the mass of the unit monopole, indicating that the attractive long-range force mediated by the Higgs field (which is massless in the BPS limit) exactly cancels the Coulomb magnetic repulsion between static monopoles. Index theory methods show that after gauge fixing there are $4n$ linearly independent normalizable zero modes about any such solution [9]. Hence, the moduli space of solutions is parameterized by $4n$ variables, which can be taken to be three position coordinates and one $U(1)$ phase for each of the component monopoles; allowing these parameters to vary with time endows the individual monopoles with linear momentum or electric charge.

Now consider the case of an arbitrary gauge group G of rank r. Its generators can be chosen to be r commuting operators H_i that span the Cartan subalgebra, together with raising and lowering operators $E_{\boldsymbol{\alpha}}$ associated with the roots $\boldsymbol{\alpha}$. The nature of the symmetry breaking is determined by the Higgs field. Its asymptotic value Φ_0 in some fixed direction can be chosen to lie in the Cartan subalgebra, thus defining a vector $\mathbf{h}$ by

$$\Phi_0 = \mathbf{h}\cdot\mathbf{H}\,. \tag{4}$$

Maximal symmetry breaking (MSB), to the subgroup $U(1)^r$, occurs if $\mathbf{h}$ has nonzero inner products with all of the root vectors. If instead there are $k \geq 1$ root vectors orthogonal to $\mathbf{h}$, then the sublattice formed by these is the root lattice for a rank k semisimple group K and there is a nonabelian unbroken symmetry (NUS) $K \times U(1)^{r-k}$.

Now recall that one can choose as a basis for the root lattice a set of r simple roots with the property that all other roots are linear combinations of these with integer coefficients all of the same sign. It turns out to be particularly convenient to choose the simple roots so that their inner products with $\mathbf{h}$ are all nonnegative. In the MSB case this uniquely determines the simple roots, which we denote by $\boldsymbol{\beta}_a$. In the NUS case we denote by $\boldsymbol{\gamma}_j$ the simple roots orthogonal to $\mathbf{h}$ (i.e., the simple roots of K) and write the remainder as $\boldsymbol{\beta}_a$. In this latter case the choice of simple roots is not unique; the various possibilities are related by the Weyl group of K.

The quantization conditions on the magnetic charge can now be easily written down. The asymptotic magnetic field must commute with Φ; hence, we can require that in the direction chosen to define Φ_0 it be of the form

$$B_i = \frac{\hat{r}_i}{4\pi r^2}\, \mathbf{g}\cdot\mathbf{H}\,. \tag{5}$$

Topological arguments [10] then imply that

$$\mathbf{g} = \frac{4\pi}{e}\left[\sum_a n_a \boldsymbol{\beta}_a^* + \sum_j q_j \boldsymbol{\gamma}_j^*\right], \tag{6}$$

where

$$\boldsymbol{\alpha}^* = \frac{\boldsymbol{\alpha}}{\boldsymbol{\alpha}^2} \tag{7}$$

is the dual of the root $\boldsymbol{\alpha}$. The integers n_a are the topologically conserved charges. For a given solution they are uniquely determined and gauge-invariant, even though the corresponding $\boldsymbol{\beta}_a$ may not be. The q_j are also integers, but are neither gauge-invariant nor conserved.

Consider first the MSB case. The energy of any BPS solution is

$$M = \mathbf{g}\cdot\mathbf{h} = \sum_a n_a \left(\frac{4\pi}{e}\mathbf{h}\cdot\boldsymbol{\beta}_a\right) \tag{8}$$

while the number of normalizable zero modes (after gauge fixing) is [11]

$$p = 4\sum_a n_a\,. \tag{9}$$

These results suggest that, in analogy with the $SU(2)$ case, all solutions might be viewed as composed of a number of fundamental monopoles, each with a single unit of topological charge, that have no internal degrees of freedom. In fact, these fundamental monopole solutions are easily constructed. Any root $\boldsymbol{\alpha}$ defines an $SU(2)$ subgroup with generators

$$\begin{aligned} t^1(\boldsymbol{\alpha}) &= \frac{1}{\sqrt{2\boldsymbol{\alpha}^2}}(E_{\boldsymbol{\alpha}} + E_{-\boldsymbol{\alpha}}) \\ t^2(\boldsymbol{\alpha}) &= -\frac{i}{\sqrt{2\boldsymbol{\alpha}^2}}(E_{\boldsymbol{\alpha}} - E_{-\boldsymbol{\alpha}}) \\ t^3(\boldsymbol{\alpha}) &= \boldsymbol{\alpha}^*\cdot\mathbf{H}\,. \end{aligned} \tag{10}$$

If $A_i^s(\mathbf{r}; v)$ and $\Phi^s(\mathbf{r}; v)$ give the $SU(2)$ solution corresponding to a Higgs expectation value v, then

$$
\begin{aligned}
A_i(\mathbf{r}) &= \sum_{s=1}^{3} A_i^s(\mathbf{r}; \mathbf{h}\cdot\boldsymbol{\beta}_a) t^s(\boldsymbol{\beta}_a) \\
\Phi(\mathbf{r}) &= \sum_{s=1}^{3} \Phi^s(\mathbf{r}; \mathbf{h}\cdot\boldsymbol{\beta}_a) t^s(\boldsymbol{\beta}_a) + (\mathbf{h} - \mathbf{h}\cdot\boldsymbol{\beta}_a^* \boldsymbol{\beta}) \cdot \mathbf{H}
\end{aligned}
\tag{11}
$$

is the fundamental monopole corresponding to the root $\boldsymbol{\beta}_a$. It carries topological charge

$$n_b = \delta_{ab} \tag{12}$$

and has a mass

$$m_a = \frac{4\pi}{e}\mathbf{h}\cdot\boldsymbol{\beta}_a^* . \tag{13}$$

Finally, there are only four zero modes about this solution, three corresponding to spatial translations and one to global rotations by the $U(1)$ generated by $\boldsymbol{\beta}_a \cdot \mathbf{H}$. (The other $r-1$ unbroken $U(1)$ factors leave the solution invariant.)

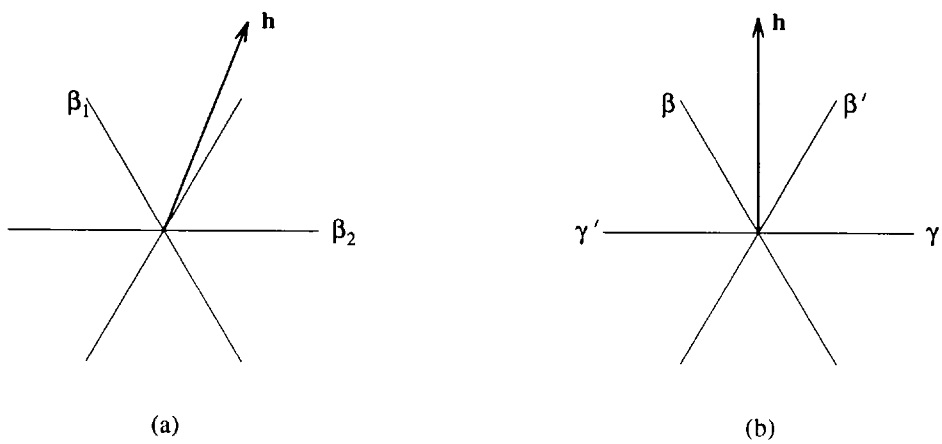

Figure 1: The root diagram of $SU(3)$. With the Higgs vector $\mathbf{h}$ oriented as in (a) the gauge symmetry is broken to $U(1) \times U(1)$, while with the orientation in (b) the breaking is to $SU(2) \times U(1)$.

As a concrete example, consider the case of $SU(3)$ broken to $U(1) \times U(1)$. With $\mathbf{h}$ as indicated in the root diagram of Fig. 1a, the the prescription described above gives the two simple roots labelled $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$. Each of these defines an embedding of the $SU(2)$ unit monopole about which there are four normalizable zero modes. One could also use the root $\boldsymbol{\beta}_1 + \boldsymbol{\beta}_2$ to define an embedding of the $SU(2)$ monopole. At first glance, one might interpret the resulting spherically symmetric solution as a third type of fundamental soliton. However, there are eight, rather than four, zero modes about this solution, indicating that it is just a special case of an eight-parameter family of two-monopole solutions. Note, for later reference, that this symmetric superposition of two fundamental monopoles has a core radius that is smaller than that of either of its components. This can be understood by noting that the presence of one monopole changes the magnitude of the vector boson mass that determines the core radius of the other.

Now let us turn to the NUS case, where an unbroken nonabelian symmetry survives. The energy of a solution, given again by Eq. (8), depends only on the n_a, not on the q_i. For technical reasons associated with the presence of long-range nonabelian fields, the index theory methods used to count zero modes can only be used if the magnetic charge is purely abelian (i.e., if $\mathbf{g} \cdot \boldsymbol{\gamma}_i = 0$ for all $\boldsymbol{\gamma}_i$); when these methods can be applied, Eq. (9) for the number of zero modes is replaced by [12]

$$p = 4 \left[\sum_a n_a + \sum_j q_j \right] . \tag{14}$$

Our experience with the MSB case might lead us to expect that the NUS solutions would also have a simple interpretation in terms of fundamental objects without internal degrees of freedom. However, there are several difficulties with this. First, while Eq. (14) suggests that there should be a fundamental monopole for each of the simple roots, the mass formula implies that those corresponding to the $\boldsymbol{\gamma}_i$ should be massless, which is impossible for a classical soliton of this theory. Second, since some of the monopole solutions obtained from the $\boldsymbol{\beta}_a$ transform nontrivially under the unbroken nonabelian factor K, one might have expected the corresponding n_a to appear with a coefficient greater than 4 in Eq. (14). Finally, because the q_j are not invariant, the number of component monopoles in a solution would depend on the choice of simple roots; a change of basis could even turn an apparently fundamental solution into a composite one.

The NUS version of the $SU(3)$ example referred to previously is shown in Fig. 1b. With $\mathbf{h}$ as indicated, the unbroken subgroup is $SU(2) \times U(1)$. The simple roots can be chosen to be either the pair labelled $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ or the pair labelled $\boldsymbol{\beta}'$ and $\boldsymbol{\gamma}'$. As in the MSB case, a solution can be obtained by embedding the $SU(2)$ monopole using the subgroup defined by $\boldsymbol{\beta}$; a gauge-equivalent solution is obtained by using $\boldsymbol{\beta}'$. Although the index methods fail, the normalizable zero modes about these two solutions can be found by direct solution of the differential equations. One finds that there are still only four such modes; the additional $SU(2)$ modes that might have been expected are nonnormalizable. Finally, as indicated above, there is no monopole solution corresponding to $\boldsymbol{\gamma}$; making the substitution $\boldsymbol{\beta}_a \to \boldsymbol{\gamma}$ in Eq. (11) gives simply the vacuum solution.

3 The moduli space approximation

The BPS multimonopole and multidyon solutions describe configurations whose component objects are all at rest relative to one another and all have the same ratio of electric to magnetic charges. One might expect that the solutions for objects with small relative velocities or arbitrary small electric charges, although not truly BPS, would in some sense be approximately BPS. This idea is formulated more precisely in the moduli space approximation [8]. Let us work in $A_0 = 0$ gauge and adopt a notation where $\Phi = A_4$ and $\partial_4 = 0$. Now let $A_a^{BPS}(\mathbf{r}, z)$ $(a = 1, \cdots, 4)$ denote a family of gauge-inequivalent static BPS solutions parameterized by n collective coordinates z_j. The moduli space approximation consists of assuming that for any t the fields can be approximated by a configuration $A_a(\mathbf{r}, t)$ that is gauge-equivalent to one of these BPS solutions; i.e.,

$$A_a(\mathbf{r},t) = U^{-1}(\mathbf{r},t)\, A_a^{BPS}(\mathbf{r}, z(t))\, U(\mathbf{r},t) - iU^{-1}(\mathbf{r},t)\, \partial_a U(\mathbf{r},t)\,. \tag{15}$$

(There is a subtle point here. Because we are working in $A_0 = 0$ gauge, only time-independent gauge transformations are at our disposal; this is why there are no time derivatives of U in Eq. (15). However, since the gauge transformation relating A_a to A_a^{BPS} might be different at one time than at another, we have to allow U to be t-dependent. Thus, $U(\mathbf{r}, t)$ should be understood as a family of static gauge transformations parameterized by t.) Given this Ansatz, the time derivatives of the fields must be of the form

$$\dot{A}_a = \dot{z}_j \frac{\partial A_a}{\partial z_j} + D_a(\dot{z}_j \epsilon_j) \equiv \dot{z}_j\, \delta_j A_a \tag{16}$$

where the gauge transformation generated by $\dot{z}_j\, \epsilon_j(\mathbf{r})$ arises from the time derivative of U. This gauge function is determined by Gauss's law,

$$0 = D_a \dot{A}_a = \dot{z}_j\, D_a(\delta_j A_a)\,. \tag{17}$$

This equation that shows the $\delta_j A_a$ are simply the background gauge zero modes about the BPS solution [13].

The $A_0 = 0$ gauge Lagrangian is

$$L = \int d^3r \,\mathrm{tr} \left[\frac{1}{2}\dot{A}_i^2 + \frac{1}{2}\dot{\Phi}^2 + \frac{1}{4}F_{ij}^2 + \frac{1}{2}D_i\Phi^2\right]. \tag{18}$$

With the Ansatz (15), the contribution of the last two terms is simply the BPS energy and therefore independent of time. Using Eq. (16) to rewrite the first two terms, we obtain

$$L = \frac{1}{2} g_{ij}(z)\, \dot{z}_i \dot{z}_j + \text{constant} \tag{19}$$

where

$$g_{ij}(z) = \int d^3x \,\mathrm{tr}\, (\delta_i A_a\; \delta_j A_a)\,. \tag{20}$$

Thus, the dynamics of the fields has been reduced to that of a point particle in geodesic motion on an n-dimensional moduli space with metric $g_{ij}(z)$.

If the full family of solutions for a given magnetic charge is known, then it is a straightforward (at least in principle) matter to obtain a complete set of background gauge zero modes and then substitute into Eq. (20) to obtain the moduli space metric. In most cases, however, an explicit solution is not available. Nevertheless, it may still be possible to determine g_{ij}. For example, the isometries of the moduli space that are implied by the space-time and internal symmetries of the theory, together with the requirement that the space be hyperkähler (this can be shown to follow from the properties of the BPS equations [14]) may be sufficient to uniquely determine the metric; this was the approach used by Atiyah and Hitchin to determine the two-monopole moduli space metric for the $SU(2)$ theory [14]. Another approach works for arbitrary magnetic charge, but only in the region of moduli space corresponding to widely separated monopoles. Here one uses the fact that the long-range interactions between monopoles are well understood, and determines the metric by the requirement that the Lagrangian (19) reproduce these interactions [15].

In particular, the moduli space and its metric are known for all cases with only two fundamental monopoles when there is maximal symmetry breaking. If the fundamental monopoles are of the same type, then the solutions are essentially embeddings of $SU(2)$ solutions and the moduli space is the Atiyah-Hitchin manifold. If the fundamental monopoles are distinct and correspond to orthogonal simple roots, then there are no interactions between the monopoles and the moduli space is simply a direct product of one-monopole moduli spaces. The final possibility is that the fundamental monopoles correspond to distinct simple roots $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ with a nonzero inner product $\lambda = -2\boldsymbol{\beta}^* \cdot \boldsymbol{\gamma}^* > 0$. As described by Kimyeong Lee in his talk at this conference [16], the moduli space for this case can be determined by a combination of the two methods described above. It is of the form [17]

$$\mathcal{M} = R^3 \times \frac{R^1 \times \mathcal{M}_0}{Z} \tag{21}$$

where the R^3 factor corresponds to the center-of-mass position and the R^1 to an overall $U(1)$ phase angle χ, while $\mathcal{M}_0$, corresponding to the relative coordinates, is the Taub-NUT space with metric

$$ds_0^2 = \left(\mu + \frac{2\pi\lambda}{e^2 r}\right) [dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)] + \left(\frac{2\pi\lambda}{e^2}\right)^2 \left(\mu + \frac{2\pi\lambda}{e^2 r}\right)^{-1} (d\psi + \cos\theta \, d\phi)^2 . \tag{22}$$

Here μ is the reduced mass of the two fundamental monopoles, r, θ, and ϕ are the relative spatial coordinates, and ψ is a relative $U(1)$ phase. The division by Z indicates that there is an identification

$$(\chi, \psi) = (\chi + 2\pi, \psi + \frac{4m_{\boldsymbol{\gamma}}}{m_{\boldsymbol{\beta}} + m_{\boldsymbol{\gamma}}}\pi) \tag{23}$$

where $m_{\boldsymbol{\beta}}$ and $m_{\boldsymbol{\gamma}}$ are the masses of the two fundamental monopoles.

In fact, one can generalize from this case to that of maximal symmetry breaking with an arbitrary number of distinct fundamental monopoles. In [18] we argued that for such moduli spaces the properties of the moduli space metric for widely separated monopoles suggested that it was in fact the exact metric over the whole moduli space. Since then, Murray and Chalmers [19] have offered proofs of this conjecture.

4 Nonabelian unbroken symmetry: an SO(5) example

As we have seen, generic values for the asymptotic Higgs field give maximal symmetry breaking, with a nonabelian unbroken symmetry emerging only for special values. This suggests that one might approach the latter case by studying the behavior of the MSB solutions as the Higgs field approaches one of these special values. However, the $SU(3)$ example illustrated in Fig. 1 shows that this limit is not always straightforward. For example, while the $\boldsymbol{\beta}_1$ monopole of the MSB case seems to have a smooth limit (the $\boldsymbol{\beta}$ monopole) as the NUS case is approached, the same is not true of the $\boldsymbol{\beta}_2$ monopole, whose mass and core radius tend to zero and infinity, respectively, in this limit. Similarly, the transition from the eight-parameter family of MSB solutions with magnetic charge proportional to $\boldsymbol{\beta}_1^* + \boldsymbol{\beta}_2^*$ to the four-parameter $\boldsymbol{\beta}'$ monopole is certainly not smooth. In all of these cases the difficulties seem to be associated with the large-distance behavior of the solutions, which is complicated by the presence of massless nonabelian gauge fields. This suggests that the NUS limit might be better behaved when the long-range magnetic fields are purely abelian; i.e., for solutions such that the $\mathbf{g} \cdot \boldsymbol{\gamma}_i$ all vanish. In particular, the MSB expressions for the mass, Eq. (8), and for the dimension of the moduli space, Eq. (9), both remain valid in the NUS limit in such cases.

The $SO(5)$ gauge theory provides a very nice example with which to test this idea. With the choice of Higgs field indicated in Fig. 2b, the symmetry is broken to $SU(2) \times U(1)$. According to Eq. (14), the minimal purely abelian magnetic charge, given by $e\mathbf{g}/4\pi = \boldsymbol{\beta}^* + \boldsymbol{\gamma}^*$, should have an eight-dimensional moduli space of solutions. It turns out that the full eight-parameter family of solutions, all of which are spherically symmetric, was explicitly found some time ago [20]. Given these solutions, it is a straightforward matter to find the moduli space metric from Eq. (20). The corresponding solutions for the maximally broken case (Fig. 2a) are composed of two fundamental monopoles. Although the explicit form of these solutions is not known, we do know their moduli space and its metric; these were given above in Eqs. (21-23). Thus, we can check by explicit calculation whether or not the NUS moduli space is indeed the expected limit of that for the MSB case.

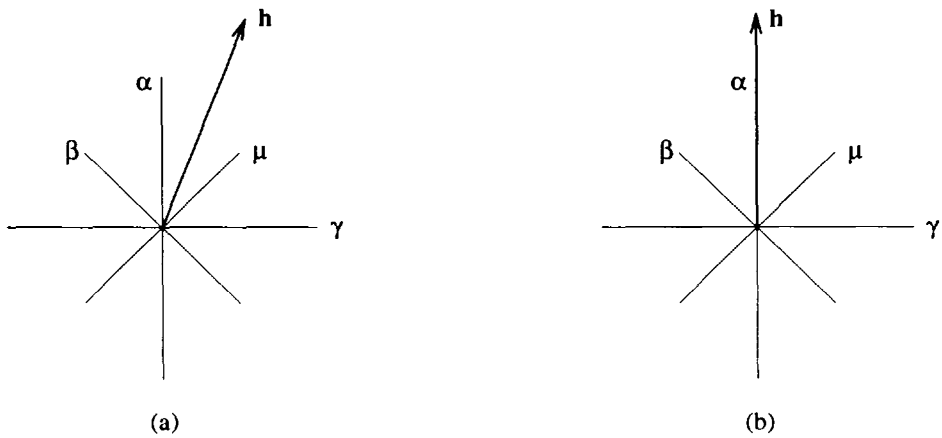

Figure 2: The root diagram of $SO(5)$. With the Higgs vector $\mathbf{h}$ oriented as in (a) the gauge symmetry is broken to $U(1) \times U(1)$, while with the orientation in (b) the breaking is to $SU(2) \times U(1)$.

Let us begin with the NUS solutions. Three of the eight parameters entering these specify the position of the center of mass, while four others are obtained by applying global $SU(2) \times U(1)$ transformations to a solution. The eighth parameter is not related to a symmetry. To see its significance we must examine the solutions in detail. First, we need some notation. Any element of P of the Lie algebra of $SO(5)$ can be expressed in terms of two vectors $\mathbf{P}_{(1)}$ and $\mathbf{P}_{(2)}$ and a 2×2 matrix $P_{(3)}$ by

$$P = \mathbf{P}_{(1)} \cdot \mathbf{t}(\boldsymbol{\alpha}) + \mathbf{P}_{(2)} \cdot \mathbf{t}(\boldsymbol{\gamma}) + \operatorname{tr} P_{(3)} M \,, \tag{24}$$

where $\mathbf{t}(\boldsymbol{\alpha})$ and $\mathbf{t}(\boldsymbol{\gamma})$ are defined as in Eq. (10) and

$$M = \frac{i}{\sqrt{\boldsymbol{\beta}^2}} \begin{pmatrix} E_{\boldsymbol{\beta}} & -E_{-\boldsymbol{\mu}} \\ E_{\boldsymbol{\mu}} & E_{-\boldsymbol{\beta}} \end{pmatrix} . \tag{25}$$

The components of $\mathbf{P}_{(1)}$ are singlets under the unbroken $SU(2)$, while $\mathbf{P}_{(2)}$ and $P_{(3)}$ transform as a triplet and a pair of doublets.

With this notation, the family of solutions found in [20] can be written as

$$\begin{aligned} A^a_{i(1)} &= \epsilon_{aim} \hat{r}_m A(r) & \phi^a_{(1)} &= \hat{r}_a H(r) \\ A^a_{i(2)} &= \epsilon_{aim} \hat{r}_m G(r,a) & \phi^a_{(2)} &= \hat{r}_a G(r,a) \\ A_{i(3)} &= \tau_i F(r,a) & \phi_{(3)} &= -iIF(r,a) \,. \end{aligned} \tag{26}$$

where $A(r)$ and $H(r)$ are the $SU(2)$ monopole functions given in Eq. (3) and

$$\begin{aligned} F(r,a) &= \frac{v}{\sqrt{8}\cosh(evr/2)} L(r,a)^{1/2} \\ G(r,a) &= A(r) L(r,a) \end{aligned} \tag{27}$$

with

$$L(r,a) = [1 + (r/a)\coth(evr/2)]^{-1} \tag{28}$$

and $v = \mathbf{h}\cdot\boldsymbol{\alpha}$.

The parameter a in these solutions has the dimensions of length and can take on any positive real value. It enters only in the doublet and triplet components of fields. Its effect on the triplet components, proportional to $G(r,a)$, is particularly striking. For $1/ev \lesssim r \lesssim a$, these fall as $1/r$, yielding the Coulomb magnetic field appropriate to a nonabelian magnetic charge. However, for larger distances the vector potential falls as $1/r^2$, showing that the magnetic charge is actually purely abelian. One might describe these solutions as being composed of a monopole core of radius $\sim 1/ev$ surrounded by a "nonabelian cloud" of radius $\sim a$.

Given these solutions, we can now obtain the moduli space metric. From symmetry considerations alone, we see that the metric must be of the form

$$\begin{aligned} ds^2 &= B(a)d\mathbf{x}^2 + C(a)d\chi^2 + I_1(a)da^2 + I_2(a)[d\theta^2 + \sin^2\theta\, d\phi^2 + (d\psi + \cos\theta\, d\phi)^2] \\ &\equiv B(a)d\mathbf{x}^2 + C(a)d\chi^2 + I_1(a)da^2 + I_2(a)\, d\Omega_3^2 \end{aligned} \tag{29}$$

where $\mathbf{x}$ is the center of mass position and χ a $U(1)$ phase angle, while the Euler angles θ, ϕ, and ψ correspond to the standard mapping of $SU(2)$ onto a three-sphere. The BPS dyon mass formula relates B and C to the monopole mass M, and shows that they are both independent of a. The remaining metric functions, I_1 and I_2, can be determined from the zero modes using Eq. (20). Making the variation $a \to a + \delta a$ in Eqs. (26) yields a zero mode that is already in background gauge and so can be directly substituted into Eq. (20) to give $g_{aa} = I_1 = 4\pi/e^2\gamma^2 a$. To obtain the $SU(2)$ zero modes, we exploit the fact that the BPS zero mode equations can be recast in the form of a Dirac equation for $\psi(x) = I\delta\phi(x) + i\sigma_j\delta A_j(x)$ [21]. Multiplication of any solution ψ on the right by a 2×2 unitary matrix yields a new solution ψ' that can be transformed back to give a new BPS zero mode, already in background gauge. In particular, if ψ_a is the Dirac solution obtained from the δa zero mode, then $\psi_a(i\hat{\mathbf{n}}\cdot\boldsymbol{\sigma})$ yields a zero mode that corresponds to a gauge transformation with a gauge function of the form $f(r)\,\hat{\mathbf{n}}\cdot\mathbf{t}(\gamma)$. One finds that $f(\infty) = 1/ea$, implying that the mode corresponding to a shift δa maps to an $SU(2)$ rotation by an angle $\delta a/a$, which in turn implies that $I_2 = a^2 I_1$. The net result is that

$$ds^2 = M d\mathbf{x}^2 + \frac{16\pi^2}{Me^4}d\chi^2 + \frac{4\pi}{e^2\gamma^2}\left[\frac{da^2}{a} + a\, d\Omega_3^2\right]. \tag{30}$$

By making the change of variables $\rho = 2\sqrt{a}$, this can be recast in the form

$$ds^2 = M d\mathbf{x}^2 + \frac{16\pi^2}{Me^4}d\chi^2 + \frac{4\pi}{e^2\gamma^2}\left[d\rho^2 + \frac{\rho^2}{4}\, d\Omega_3^2\right]. \tag{31}$$

The quantity in brackets is the metric for R^4 written in polar coordinates (with the factor of $1/4$ arising from the normalization of the Euler angles), and so the moduli space is the flat manifold

$$\mathcal{M} = R^3 \times S^1 \times R^4 \tag{32}$$

with the standard metric. (The second factor is S^1 because of the periodicity of χ.)

The corresponding MSB moduli space was described in the previous section. The NUS case corresponds to the limit in which $m_{\boldsymbol{\gamma}}$, and hence the reduced mass μ, both vanish. In this limit the identification (23) reduces to $(\chi, \psi) = (\chi + 2\pi, \psi)$. The division by Z thus acts only on the R^1 factor, allowing us to write

$$\mathcal{M} = R^3 \times S^1 \times \mathcal{M}_0 \,. \tag{33}$$

Furthermore, if we set $\mu = 0$ and use the fact that $\lambda = -2\boldsymbol{\beta}^* \cdot \boldsymbol{\gamma}^* = 2/\boldsymbol{\gamma}^2$, the metric (22) for $\mathcal{M}_0$ becomes

$$ds_0^2 = \frac{4\pi}{e^2\boldsymbol{\gamma}^2 r}\left(dr^2 + r^2\, d\Omega_3^2\right) \,. \tag{34}$$

Comparing these results with Eqs. (30) and (32), we see that the NUS moduli space metric is indeed the $m_{\boldsymbol{\gamma}} \to 0$ limit of that for the MSB case, provided that intermonopole distance r is identified with the cloud radius a.

Although the moduli space behaves smoothly as one case goes over into the other, there is a curious change in the interpretation of its coordinates. The intermonopole distance becomes the radius of the nonabelian cloud, while the angles specifying the relative spatial orientation of the the two monopoles combine with their relative $U(1)$ phase to give the global $SU(2)$ orientation of the solution. Thus, the degrees of freedom of the $\boldsymbol{\gamma}$ monopole remain, but they are no longer attributable to an isolated object. Instead, they describe the properties of a cloud that surrounds the $\boldsymbol{\beta}$ monopole and cancels the nonabelian magnetic charge that the latter acquires in the NUS limit.

We can see quite clearly how this comes about if we follow the behavior of a generic MSB solution as the NUS limit is approached. Thus, let us start with a two-monopole configuration in which the intermonopole distance r is much greater than the core radii $R_{\boldsymbol{\beta}} \sim (e^2 m_{\boldsymbol{\beta}})^{-1}$ and $R_{\boldsymbol{\gamma}} \sim (e^2 m_{\boldsymbol{\gamma}})^{-1}$ of the two monopoles. We can then take the NUS limit by varying the asymptotic Higgs field so that $m_{\boldsymbol{\gamma}} \to 0$ while $m_{\boldsymbol{\beta}}$ remains fixed. If we had only a $\boldsymbol{\gamma}$ monopole, its core would expand without limit as we did this. However, in the presence of the second monopole, the $\boldsymbol{\gamma}$ core only expands until it reaches the $\boldsymbol{\beta}$-monopole, at which point it begins to evolve into a spherical cloud whose size is set by the original intermonopole distance. The reduced radius of the $\boldsymbol{\gamma}$-monopole in the presence of a nearby $\boldsymbol{\beta}$-monopole can be understood in the same manner as that for the spherically symmetric superposition of a $\boldsymbol{\beta}_1$ and a $\boldsymbol{\beta}_2$ $SU(3)$ monopole that was noted previously.

5 More complex examples

In the $SO(5)$ example we saw how the two-monopole MSB solutions evolved into spherically symmetric solutions with one massive and one massless monopole in the limit of nonabelian symmetry breaking. Because the exact NUS solutions were known, it was possible to verify explicitly that the NUS moduli space was the expected limit of that for the maximally broken case. I will now discuss some more complex examples.

The first of these, with two massive monopoles in the NUS limit, arises in a theory with gauge group $SU(4)$. The solutions are not spherically symmetric, even in the NUS limit, and so it not surprising that their explicit forms are not known. However, because we have the MSB moduli space metric, we check that it has the required isometries in the NUS limit.

In this example the symmetry is broken to $U(1) \times SU(2) \times U(1)$, with the $SU(2)$ factor corresponding to the middle root of the Dynkin diagram in Fig. 3. (This corresponds to a Higgs field expectation value of the form $\Phi = \mathrm{diag}\,(\phi_1, \phi_2, \phi_2, \phi_3)$ with $\phi_1 > \phi_2 > \phi_3$.) If

$$\mathbf{g} = \frac{4\pi}{e}(\boldsymbol{\beta}_1^* + \boldsymbol{\gamma}^* + \boldsymbol{\beta}_2^*)\,, \tag{35}$$

then (1) $\mathbf{g} \cdot \boldsymbol{\gamma} = 0$, implying that the asymptotic magnetic field is purely abelian and that the moduli space should have a smooth NUS limit, and (2) there is only one fundamental monopole of each type, so the MSB moduli space is of the class for which the metric was given in [18]. Examination of this metric shows that for the generic MSB case it has an $SU(2)$ isometry corresponding to spatial rotations and a $U(1)^3$ isometry corresponding to the unbroken gauge symmetry. However, the generator of one of these $U(1)$ factors combines with two other vector fields, both of which become Killing vectors in the NUS limit, to generate an $SU(2)$, so that in the NUS limit the $U(1)^3$ isometry is enlarged to the required $U(1) \times SU(2) \times U(1)$.

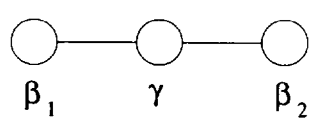

Figure 3: The Dynkin diagram of $SU(4)$, with the labelling of the simple roots corresponding to symmetry breaking to $U(1) \times SU(2) \times U(1)$.

The moduli space has twelve dimensions. For the MSB case, a natural choice of coordinates is given by three position variables and one $U(1)$ phase for each of the three monopoles. In the NUS limit, the cores of the $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ monopoles remain finite and (assuming that they are not too close) distinct. Hence, the six position variables specifying their locations should continue to be good moduli space coordinates, as should their $U(1)$ phases. The overall $SU(2)$ orientation of the solution gives three more coordinates. This leaves just one other variable which, with the $SO(5)$ example in mind, we might expect to in some way characterize a nonabelian cloud surrounding the massive monopoles.

We saw in the $SO(5)$ example that the MSB intermonopole distance became in the NUS limit a natural choice for an $SU(2)$-invariant parameter describing the nonabelian cloud. A nice generalization of this occurs here. Let r_1 and r_2 be the distances from the $\boldsymbol{\gamma}$ monopole to the $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ monopoles, respectively. Their sum, $a = r_1 + r_2$, (but not r_1 or r_2 separately) is left invariant by the vector fields that generate the $SU(2)$ isometry and can be taken as the twelfth moduli space coordinate. Indeed, under the $SU(2)$ transformations generated by these vector fields the orbit of the "position" of the $\boldsymbol{\gamma}$ monopole is the ellipsoid $r_1 + r_2 = a$.

This example can be generalized to the case of $SU(N+2)$ broken to $U(1) \times SU(N) \times U(1)$. With the roots labelled as in Fig. 4, the magnetic charge

$$\frac{e\mathbf{g}}{4\pi} = \boldsymbol{\beta}_1^* + \sum_{j=2}^{N} \boldsymbol{\gamma}_j^* + \boldsymbol{\beta}_{N+1}^* \tag{36}$$

is orthogonal to the $\boldsymbol{\gamma}_j$'s that span the unbroken $SU(N)$. Because there is only one fundamental monopole of each type, the MSB metric of [18] is again applicable. Examining this metric, one finds that there is a set of vector fields, generating a $U(1) \times SU(N) \times U(1)$ algebra, that gives the required isometry in the NUS limit. The dimension of the moduli space is $4(N+1)$. As usual, the coordinates for maximal symmetry breaking can be taken to be three position and one $U(1)$ variable for each of the $N+1$ fundamental monopoles. With the nonabelian breaking, the positions and $U(1)$ phases of the two monopoles that remain massive give eight parameters. The action of the unbroken $SU(N)$ gives additional parameters; because a generic solution is left invariant by a $U(N-2)$ subgroup, there are only $[\dim SU(N) - \dim U(N-2)] = 4N-5$ of these. This leaves just one moduli space coordinate to be specified; an $SU(N)$-invariant choice for this is $a = \sum_1^N r_j$, where the r_j are the distances between fundamental monopoles whose corresponding simple roots are linked in the Dynkin diagram of Fig. 4.

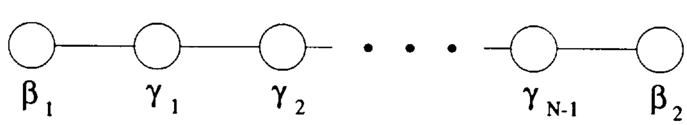

Figure 4: The Dynkin diagram of $SU(N+2)$, with the labelling of the simple roots corresponding to symmetry breaking to $U(1) \times SU(N) \times U(1)$.

A purely abelian asymptotic field is also obtained if

$$\frac{e\mathbf{g}}{4\pi} = N\boldsymbol{\beta}_1^* + \sum_{j=1}^{j=N-1} (N-j)\boldsymbol{\gamma}_{j+1}^* \, . \tag{37}$$

In the NUS limit this corresponds to a family of solutions containing N massive and $N(N-2)/2$ massless monopoles, with a moduli space of dimension $2N(N+1)$. The positions and $U(1)$ phases of the massive monopoles give $4N$ of the coordinates, while the overall $SU(N)$ orientation gives N^2-1 more. (The generic solution has no invariance subgroup.) This leaves $(N-1)^2$ parameters that describe the gauge-invariant properties of the nonabelian cloud, which evidently can have a much more complex structure than it did in the previous examples.

The existence of two independent "color-neutral" magnetic charges for this symmetry breaking (all others are linear combinations of these two) is easily understood. The massive monopoles corresponding to $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ are objects that transform under the fundamental representations F and $\bar{F}$ of the unbroken group, respectively. The combinations given above correspond to the fact that one can obtain an $SU(N)$ singlet either with an F and an $\bar{F}$, or from N F's.

6 Concluding remarks

In this talk I have shown how one can study monopoles with nonabelian magnetic charges by following the behavior of purely abelian monopoles as the asymptotic Higgs field is varied toward one of the special values that leaves an nonabelian subgroup of the gauge symmetry unbroken. In the limit of nonabelian breaking, some of the abelian monopoles remain massive, but acquire nonabelian components to their magnetic charge. Others, whose mass tends to zero in this limit, evolve into a cloud that surrounds one or more massive monopoles and cancels their nonabelian magnetic charge. Although they cease to exist as distinct objects, their degrees of freedom survive as parameters describing this cloud.

The analysis described here has been largely classical. The effects of quantum corrections remain to be investigated. One would want to see, for example, the effects of confinement on the nonabelian magnetic charges. It would also be desirable to go beyond the semiclassical approximation and make contact with the results of Seiberg and Witten [22].

Perhaps most intriguing are the issues related to the duality conjecture. The monopoles that become massless in the limit of nonabelian breaking are presumably the duals to the massless gauge bosons. A fuller understanding of these objects might well form the basis for an approach to nonabelian interactions complementary to that based on the perturbative gauge bosons.

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